BANACH-STONE THEOREMS FOR MAPS PRESERVING COMMON ZEROS

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ABSTRACT. Let X and Y be completely regular spaces and E and F be Hausdorff topological vector spaces. We call a linear map T from a subspace of $C(X, E)$ into $C(Y, F)$ a *Banach-Stone map* if it has the form $Tf(y) = S_y(f(h(y)))$ for a family of linear operators $S_y : E \to F$, $y \in Y$, and a function $h : Y \to X$. In this paper, we consider maps having the property:

(Z) $$\cap_{i=1}^k Z(f_i) \neq \emptyset \iff \cap_{i=1}^k Z(Tf_i) \neq \emptyset,$$

where $Z(f) = \{f = 0\}$. We characterize linear bijections with property (Z) between spaces of continuous functions, respectively, spaces of differentiable functions (including C^∞), as Banach-Stone maps. In particular, we confirm a conjecture of Ercan and Önal:

Suppose that X and Y are realcompact spaces and E and F are Hausdorff topological vector lattices (respectively, C^*-algebras). Let $T : C(X, E) \to C(Y, F)$ be a vector lattice isomorphism (respectively, $*$-algebra isomorphism) such that

$$Z(f) \neq \emptyset \iff Z(Tf) \neq \emptyset.$$

Then X is homeomorphic to Y and E is lattice isomorphic (respectively, C^*-isomorphic) to F.

Some results concerning the continuity of T are also obtained.

1. INTRODUCTION

Let X and Y be compact Hausdorff spaces. A linear map $T : C(X) \to C(Y)$ is a surjective isometry if and only if it has the form $Tf = w \cdot (f \circ h)$ for some homeomorphism $h : Y \to X$ and some function $w \in C(Y)$ such that $|w(y)| = 1$ for all y. This classical result, called the Banach-Stone Theorem, was obtained by Banach [6] for compact metric spaces and extended by Stone [21] to compact Hausdorff spaces. Variations of this result were obtained by Gelfand and Kolmogorov [15] for algebraic isomorphisms and Kaplansky [17] for vector lattice isomorphisms. There has been a great deal of development in this area in the intervening period, including extensions to spaces of vector-valued functions. Let us mention in particular the work on Behrends [7] on M-structures and the Banach-Stone property and the theory of separating and biseparating maps (see, e.g., [1, 2, 3, 4, 5, 13, 16, 19]).

We refer to the article [12] for a survey on the many results and research areas arising from the Banach-Stone Theorem.

In a recent article, Ercan and Önal obtained a Banach lattice-valued version of the Banach-Stone Theorem.

Theorem 1. (Ercan and Önal [10]) *Let X and Y be compact Hausdorff spaces and E and F be Banach lattices. If there is an onto vector lattice isomorphism $T : C(X, E) \to C(Y, F)$ such that*

$$(\mathrm{Z}_0) \qquad Z(f) = \{f = 0\} \neq \emptyset \iff Z(Tf) \neq \emptyset,$$

and F is an AM-space with unit, then X is homeomorphic to Y and E and F are isomorphic as Banach lattices.

They further conjectured that the result holds for any Banach lattice F. It is easily observed that for a vector lattice isomorphism, condition (Z_0) is equivalent to

$$(\mathrm{Z}) \qquad \cap_{i=1}^{k} Z(f_i) \neq \emptyset \iff \cap_{i=1}^{k} Z(Tf_i) \neq \emptyset$$

for any finite collection of functions $(f_i)_{i=1}^{k}$. Maps satisfying (Z) are termed *maps preserving common zeros.* In this paper, we undertake a general investigation of linear operators preserving common zeros mapping between certain spaces of vector-valued functions. As a result, we are able to confirm the conjecture of Ercan and Önal in a general setting. We also obtain Banach-Stone type results for mappings between spaces of vector-valued differentiable functions and some theorems on the automatic continuity of such maps. Professor N. C. Wong has informed us that he and his co-authors have independently solved the conjecture of Ercan and Önal [8]. The conjecture was also solved by its proposers Ercan and Önal [11].

In §2, there is a study of maps preserving common zeros under rather general conditions. Theorem 6 gives a description of such maps, which shows that they are almost of Banach-Stone form. In §3, we specialize to the case of vector-valued continuous functions, particularly on realcompact spaces. In this case, Theorem 10 provides a complete characterization of maps preserving common zeros. The conjecture of Ercan and Önal [10] follows as a corollary. In §4, we study maps preserving common zeros mapping between spaces of vector-valued differentiable functions. We are able to characterize such maps as Banach-Stone maps under the general assumption that E and F are Hausdorff topological vector spaces (Theorem 16). Our result also holds for C^{∞}-functions. In the final section, we investigate the continuity properties of the maps considered in §4. Even without assuming completeness of E and F, continuity of the associated map Φ can be obtained for maps between C^m-spaces when $m \in \mathbb{N}$. However, this is not longer true if $m = \infty$. To obtain full continuity of the map T requires completeness of E and F. These results, summarized in Theorem 23 and Examples 24 and 27, serve to clarify the role played by completeness in theorems regarding automatic continuity.

2. Maps preserving common zeros

If X is a (Hausdorff) completely regular space and E is a Hausdorff topological vector space, let $C(X,E)$ be the set of all continuous functions from X into E. It is known that E is completely regular [20]. In particular, every $f \in C(X,E)$ has a unique continuous extension $f^\beta : \beta X \to \beta E$, where βX and βE are the Stone-Čech compactifications of X and E respectively. The space $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$, as the case may be, is abbreviated to $C(X)$. A vector subspace $A(X)$ of $C(X)$ is said to be *almost normal* if for every pair of subsets P and Q of X such that $\overline{P}^{\beta X} \cap \overline{Q}^{\beta X} = \emptyset$, there exists f in $A(X)$ such that $f(P) \subseteq \{0\}$ and $f(Q) \subseteq \{1\}$. A vector subspace $A(X,E)$ of $C(X,E)$ is said to be *almost normally multiplicative* if $A(X,E)$ contains the constant functions and there is an almost normal subspace $A(X)$ of $C(X)$ so that $\varphi \cdot f \in A(X,E)$ whenever $\varphi \in A(X)$ and $f \in A(X,E)$. For any $f \in C(X,E)$, let $Z(f) = \{x \in X : f(x) = 0\}$ be the *zero set of* f. If $u \in E$, let $\mathbf{u}$ be the function in $C(X,E)$ with constant value u.

In this paper, X and Y will always denote (Hausdorff) completely regular spaces and E and F (nontrivial) Hausdorff topological vector spaces. If $A(X,E)$ and $A(Y,F)$ are subspaces of $C(X,E)$ and $C(Y,F)$ respectively, a linear map $T : A(X,E) \to A(Y,F)$ is said to *preserve common zeros* if for any $k \in \mathbb{N}$ and any sequence $(f_i)_{i=1}^k$ in $A(X,E)$,

(Z) $$\cap_{i=1}^k Z(f_i) \neq \emptyset \Longleftrightarrow \cap_{i=1}^k Z(Tf_i) \neq \emptyset.$$

For the rest of the section, $A(X,E)$ and $A(Y,F)$ will denote almost normally multiplicative vector subspaces of $C(X,E)$ and $C(Y,F)$ respectively and $A(X)$ and $A(Y)$ are the corresponding almost normal subspaces of $C(X)$ and $C(Y)$.

Proposition 2. *Suppose that $T : A(X,E) \to A(Y,F)$ is a surjective linear map that preserves common zeros. Then there exist a dense subset Z of βY and a homeomorphism $h : Z \to X$ so that, for all $y \in Z$ and all $f \in A(X,E)$, there is a net (y_α) in Y converging to y such that*

(1) $$(Tf)(y_\alpha) = (T\mathbf{u})(y_\alpha) \text{ for all } \alpha, \text{ where } u = f(h(y)).$$

In particular, $(Tf)^\beta(y) = (T\mathbf{u})^\beta(y)$.

By (Z) and the compactness of βY, the set

$$Z_x = \bigcap_{\substack{f \in A(X,E) \\ f(x)=0}} \overline{Z(Tf)}^{\beta Y}$$

is nonempty for all $x \in X$. Before giving the proof of Proposition 2, we first establish a number of lemmas.

Lemma 3. *If $y \in Z_x$, then for all $f \in A(X,E)$, there exists a net $(y_\alpha) \subseteq Y$ converging to y such that*

$$(Tf)(y_\alpha) = (T\mathbf{u})(y_\alpha) \text{ for all } \alpha,$$

where $u = f(x)$. As a result,

$$(Tf)^\beta (y) = (T\mathbf{u})^\beta (y).$$

Proof. For any $f \in A(X, E)$, $(f - \mathbf{u})(x) = 0$, where $u = f(x)$. Hence $y \in Z_x \subseteq \overline{Z(T(f - \mathbf{u}))}^{\beta Y}$. Thus there exists $(y_\alpha) \subseteq Z(T(f - \mathbf{u}))$ converging to y. Then

$$(Tf)(y_\alpha) = (T\mathbf{u})(y_\alpha) \text{ for all } \alpha.$$

Taking limits yield

$$(Tf)^\beta (y) = (T\mathbf{u})^\beta (y).$$

$\square$

Lemma 4. *If $x_1 \neq x_2$, then $Z_{x_1} \cap Z_{x_2} = \emptyset$.*

Proof. Suppose that $y_0 \in Z_{x_1} \cap Z_{x_2}$. Take any $f \in A(X, E)$ and let $u = f(x_1)$. Since $y_0 \in Z_{x_1}$, by Lemma 3,

$$(Tf)^\beta (y_0) = (T\mathbf{u})^\beta (y_0).$$

There exists $\varphi \in A(X)$ such that $\varphi(x_1) = 1$ and $\varphi(x_2) = 0$. Since $\varphi f \in A(X, E)$, we may apply Lemma 3 to φf (at x_1) to obtain

$$(T\mathbf{u})^\beta (y_0) = (T(\varphi f))^\beta (y_0).$$

However, since $y_0 \in Z_{x_2}$, by Lemma 3 yet again, $(T(\varphi f))^\beta (y_0) = (T\mathbf{v})^\beta (y_0)$, where $v = \varphi(x_2) f(x_2) = 0$. Thus

$$(Tf)^\beta (y_0) = (T\mathbf{v})^\beta (y_0) = 0.$$

This contradicts the surjectivity of T, since $\mathbf{w} \in A(Y, F)$ for any $w \in F \setminus \{0\}$. $\square$

Lemma 5. $|Z_x| = 1$ *for all $x \in X$.*

Proof. Suppose, to the contrary, that there are distinct $y_0, y_1 \in Z_x$. Let U_0, U_1 be open neighborhoods of y_0 and y_1 in βY respectively whose closures are disjoint. By almost normality of $A(Y)$, there exists $\varphi \in A(Y)$ such that

$$\varphi = 0 \text{ on } U_0 \cap Y \text{ and } \varphi = 1 \text{ on } U_1 \cap Y.$$

Let $v \in F \setminus \{0\}$ and $g = \varphi \cdot \mathbf{v} \in A(Y, F)$. From the fact that T is onto, there exists $f \in A(X, E)$ such that $Tf = g$. By Lemma 3, for $i = 0, 1$, there exist (y_α^i) in Y converging to y_i such that

$$(T\mathbf{u})(y_\alpha^i) = (Tf)(y_\alpha^i) = \varphi(y_\alpha^i) v, \text{ where } u = f(x).$$

Since $\varphi(y_\alpha^0) = 0$ for sufficiently large α, $(T\mathbf{u})(y_\alpha^0) = 0$ for such α's. It follows that $Z(T\mathbf{u}) \neq \emptyset$. By (Z), $Z(\mathbf{u}) \neq \emptyset$ and hence $f(x) = u = 0$. On the other hand, $\lim_\alpha \varphi(y_\alpha^1) v = v \neq 0$. But

$$\varphi(y_\alpha^1) v = (Tf)(y_\alpha^1) = (T\mathbf{u})(y_\alpha^1) = (T\mathbf{0})(y_\alpha^1) = 0,$$

a contradiction. $\square$

Proof of Proposition 2. Let $Z = \bigcup_x Z_x$. Then $Z \subseteq \beta Y$. Define a mapping $h : Z \to X$ by sending the unique element in Z_x to x. The mapping h is well-defined by Lemma 4. Clearly h is a bijection and for any $y \in Z$ and $f \in A(X, E)$, Lemma 3 yields a net (y_α) in Y converging to y so that (1) is satisfied. It remains to show that h and h^{-1} are continuous, and that Z is dense in βY.

Suppose that h is not continuous. Using the compactness of βX, there exists $y_0 \in Z$, $x' \in \beta X$ and a net (y_α) in Z converging to y_0 such that

$$h(y_\alpha) = x_\alpha \to x' \neq x_0 = h(y_0).$$

Let U be an open neighborhood of x' in βX such that $\overline{U}^{\beta X}$ does not contain x_0. Choose $\varphi \in A(X)$ such that $\varphi = 0$ in $U \cap X$ and $\varphi(x_0) = 1$. For all $f \in A(X, E)$. We have by Lemma 3 that

$$(T(\varphi f))(y_\alpha) = (T\mathbf{u}_\alpha)(y_\alpha) \text{ for all } \alpha,$$

where $u_\alpha = \varphi(x_\alpha) f(x_\alpha)$. Since $u_\alpha = \varphi(x_\alpha) f(x_\alpha) = 0$ for sufficiently large α, $(T(\varphi f))^\beta (y_0) = 0$. On the other hand, $(T(\varphi f))^\beta (y_0) = (T\mathbf{v})^\beta (y_0)$, where $v = \varphi(x_0) f(x_0) = f(x_0)$. Thus $(Tf)^\beta (y_0) = 0$ for all $f \in A(X, E)$, contradicting the surjectivity of T.

Suppose that h^{-1} is not continuous. Using the compactness of βY, there exists $x_0 \in X$, $y' \in \beta Y$ and a net (x_α) in X converging to x_0 such that $h^{-1}(x_\alpha) = y_\alpha \to y' \neq y_0 = h^{-1}(x_0)$. Let U and V be open neighborhoods of y_0 and y' in βY respectively whose closures are disjoint. Choose $\varphi \in A(Y)$ such that $\varphi = 1$ in $U \cap Y$ and $\varphi = 0$ on $V \cap Y$. Let $v \in F \smallsetminus \{0\}$ and set $g = \varphi \cdot \mathbf{v} \in A(Y, F)$. Since T is surjective, there is an $f \in A(X, E)$ such that $Tf = g$. By Lemma 3, for each α, there exists a net $(y_\alpha^\gamma)_\gamma$ in Y converging to y_α such that

$$(T\mathbf{u}_\alpha)(y_\alpha^\gamma) = (Tf)(y_\alpha^\gamma) \text{ for all } \gamma, \text{ where } u_\alpha = f(x_\alpha),$$
$$= g(y_\alpha^\gamma) = \varphi(y_\alpha^\gamma) v \text{ for all } \gamma.$$

For sufficiently large α, there exists γ where $\varphi(y_\alpha^\gamma) = 0$. Hence $Z(T\mathbf{u}_\alpha) \neq \emptyset$ for large enough α. By (Z), $Z(\mathbf{u}_\alpha) \neq \emptyset$ and consequently, $f(x_\alpha) = 0$. Thus $f(x_0) = 0$. Therefore, by Lemma 3,

$$0 = (T\mathbf{0})^\beta (y_0) = (Tf)^\beta (y_0)$$
$$= (\varphi \cdot \mathbf{v})^\beta (y_0)$$
$$= \varphi^\beta (y_0) v$$
$$= v \neq 0,$$

a contradiction.

Finally, we show that Z is dense in βY. If this is not true, then there exists $y_0 \in \beta Y \smallsetminus \overline{Z}^{\beta Y}$. Let U and V be open neighborhoods of $\overline{Z}^{\beta Y}$ and y_0 in βY respectively whose closures are disjoint. There exists $\varphi \in A(Y)$ so that $\varphi = 0$ on $U \cap Y$ and $\varphi = 1$ on $V \cap Y$. Take $v \in F \smallsetminus \{0\}$ and set

$g = \varphi \cdot \mathbf{v} \in A\left(Y, F\right).$ By the surjectivity of T, there exists $f \in A\left(X, E\right)$ such that $Tf = g$. Let $x \in X$ and $y \in Z_x \subseteq U$. By Lemma 3, there exists $(y_\alpha) \subseteq Y$ converging to y such that

$$(T\mathbf{u})\left(y_\alpha\right) = (Tf)\left(y_\alpha\right) \text{ for all } \alpha, \text{ where } u = f\left(x\right)$$
$$= \varphi\left(y_\alpha\right)v.$$

For sufficiently large α, $\varphi\left(y_\alpha\right) = 0$. Thus $Z\left(T\mathbf{u}\right) \neq \emptyset$ and by (Z), $Z\left(\mathbf{u}\right) \neq \emptyset$, which means that $u = 0$. Since x is arbitrary, $f = 0$. Thus $g = 0$, which is absurd. $\square$

The next theorem, which is the main result of the section, gives a preliminary description of vector space isomorphisms preserving common zeros. Applications of this theorem in certain cases yield sharp characterizations of said mappings. These applications will be our concern in the subsequent sections.

Theorem 6. *Suppose that $T : A(X, E) \to A(Y, F)$ is a vector space isomorphism that preserves common zeros. Then there are dense subsets Z of βY and W of βX and homeomorphisms $h : Z \to X$, $k : W \to Y$ so that $h \cup k^{-1} : Z \cup Y \to X \cup W$ is a homeomorphism. Moreover, for all $f \in A(X, E)$, $y \in Z$,*

$$(Tf)^\beta(y) = (T\mathbf{u})^\beta(y), \ u = f(h(y)),$$

and for all $g \in A(Y, F)$, $x \in W$,

$$(T^{-1}g)^\beta(x) = (T^{-1}\mathbf{v})^\beta(x), \ v = g(k(x)).$$

Proof. Applying Proposition 2 to both T and T^{-1}, we find dense subsets Z of βY, W of βX and homeomorphisms $h : Z \to X$, $k : W \to Y$ such that

$$(2) \qquad\qquad (Tf)^\beta(y) = (T\mathbf{u})^\beta(y), \ u = f(h(y))$$

whenever $f \in A(X, E)$, $y \in Z$ and

$$(3) \qquad\qquad (T^{-1}g)^\beta(x) = (T^{-1}\mathbf{v})^\beta(x), \ v = g(k(x))$$

whenever $g \in A\left(Y, F\right)$, $x \in W$.

In the notation of Lemmas 7 and 8, $\tilde{h}_{|Z\cup Y} = \tilde{h}_{|Z} \cup \tilde{h}_{|Y} = h \cup k^{-1}$ is a well-defined map from $Z \cup Y$ to $X \cup W$. By symmetry, $h^{-1} \cup k$ is also a well-defined continuous map. Therefore, $h \cup k^{-1}$ is a homeomorphism. The proof of Theorem 6 is complete. $\square$

Lemma 7. *There is a continuous extension $\tilde{h} : \beta Y \to \beta X$ of $h : Z \to X$.*

Proof. If the lemma fails, then there exist $\left(y_\alpha^1\right), \left(y_\alpha^2\right)$ in Z converging to $y_0 \in \beta Y$ such that $h\left(y_\alpha^i\right) \to x_i \in \beta X$, $i = 1, 2$, with $x_1 \neq x_2$. Let U and V be open neighborhoods of x_1 and x_2 in βX respectively with disjoint closures. There exists $\varphi \in A\left(X\right)$ such that $\varphi = 0$ on $U \cap X$ and $\varphi = 1$ on $V \cap X$. Let $v \in F \smallsetminus \{0\}$ and pick $f \in A\left(X, E\right)$ such that $Tf = \mathbf{v}$. Applying (2), we find that for all α,

$$v = (Tf)^\beta \left(y_\alpha^2\right) = \left(T\mathbf{w}_\alpha^2\right)^\beta \left(y_\alpha^2\right), \text{ where } w_\alpha^2 = f\left(h\left(y_\alpha^2\right)\right)$$

and

$$\left(T\left(\varphi f\right)\right)^\beta \left(y_\alpha^i\right) = \left(T\mathbf{u}_\alpha^i\right)^\beta \left(y_\alpha^i\right), \text{ where } u_\alpha^i = \left(\varphi f\right)\left(h\left(y_\alpha^i\right)\right), \ i = 1, 2.$$

For sufficiently large α, $\varphi\left(h\left(y_\alpha^1\right)\right) = 0$ and $\varphi\left(h\left(y_\alpha^2\right)\right) = 1$ and thus

$$u_\alpha^i = \begin{cases} 0 & \text{if } i = 1, \\ w_\alpha^2 & \text{if } i = 2. \end{cases}$$

It follows that for such α,

$$\left(T\left(\varphi f\right)\right)^\beta \left(y_\alpha^i\right) = \begin{cases} 0 & \text{if } i = 1, \\ (Tf)^\beta \left(y_\alpha^2\right) & \text{if } i = 2, \end{cases}$$

$$= \begin{cases} 0 & \text{if } i = 1, \\ v & \text{if } i = 2. \end{cases}$$

Upon taking limits, we have $v = \left(T\left(\varphi f\right)\right)^\beta \left(y_0\right) = 0$, a contradiction. $\square$

Lemma 8. $\tilde{h}_{|Y} = k^{-1}$.

Proof. Suppose that there exists $y_0 \in Y$ such that $x' = \tilde{h}\left(y_0\right)$ is different from $x_0 = k^{-1}\left(y_0\right)$. Let (y_α) be a net in Z converging to y_0 and let $x_\alpha = \tilde{h}\left(y_\alpha\right) = h\left(y_\alpha\right)$. Then $x_\alpha \to x'$. Let U and V be respective open neighborhoods of x' and x_0 in βX with disjoint closures. Choose $\varphi \in A\left(X\right)$ such that $\varphi = 0$ on $U \cap X$ and $\varphi = 1$ on $V \cap X$ and set $f = \varphi \cdot \mathbf{u}$ for some fixed $u \in E \smallsetminus \{0\}$. For all α,

$$(Tf)^\beta \left(y_\alpha\right) = \left(T\mathbf{u}_\alpha\right)^\beta \left(y_\alpha\right), \text{ where } u_\alpha = f\left(x_\alpha\right)$$
$$= 0 \text{ when } \alpha \text{ is sufficiently large.}$$

Thus $0 = (Tf)^\beta \left(y_0\right) = (Tf)\left(y_0\right)$, since $y_0 \in Y$. It follows by (3) that

$$f^\beta \left(x_0\right) = \left(T^{-1}\left(Tf\right)\right)^\beta \left(x_0\right) = \left(T^{-1}\mathbf{0}\right)^\beta \left(x_0\right) = 0,$$

contradicting the fact that $f(x) = u \neq 0$ for $x \in V \cap X$. $\square$

3. Spaces of continuous functions

In this section, we apply the results of the previous section to the case where $A(X, E) = C(X, E)$ and $A(Y, F) = C(Y, F)$. Obviously, these spaces are almost normally multiplicative. Since the one-point compactification $\mathbb{R}_\infty$ of $\mathbb{R}$ is compact, every $f \in C(X)$ has a unique extension to a continuous function $f^* : \beta X \to \mathbb{R}_\infty$. Recall that the *Hewitt realcompactification* υX of X [14] is the set

$$\{x \in \beta X : f^*(x) \in \mathbb{R} \text{ for all } f \in C(X)\}.$$

X is said to be *realcompact* if $X = \upsilon X$.

Proposition 9. *Let $A(X, E) = C(X, E)$ and $A(Y, F) = C(Y, F)$ in Proposition 2. Then the set Z obtained in that Proposition is a subset of υY.*

Proof. Suppose, to the contrary, that there exists $y_0 \in Z \smallsetminus vY$. Then there exists $\psi \in C(Y)$ such that $\psi(y) \geq 1$ for all $y \in Y$ and $\psi^*(y_0) = \infty$. Clearly $\psi^*(y) \neq 0$ for all $y \in \beta Y$. Define $\varphi : X \to \mathbb{R}$ by

$$\varphi(x) = \frac{1}{\psi^*(h^{-1}(x))} \quad \left(\text{taking } \frac{1}{\infty} = 0\right).$$

Then $\varphi \in C(X)$. Let $v \in F \smallsetminus \{0\}$ and choose $f \in C(X, E)$ such that $Tf = \psi \cdot \mathbf{v}$.

Claim 1. If $y \in Z$ and $\psi^*(y) \in \mathbb{R}$, then $(T(\varphi f))^\beta(y) = v$.
Proof of Claim 1. Suppose that $y \in Z$ and $\psi^*(y) \in \mathbb{R}$. Let $c = \varphi(h(y))$ and $u = f(h(y))$. By Proposition 2,

$$\begin{aligned}
(T\mathbf{u})^\beta(y) &= (Tf)^\beta(y) \\
&= (\psi \cdot \mathbf{v})^\beta(y) \\
&= \psi^*(y) v.
\end{aligned}$$

Since $c\psi^*(y) = \varphi(h(y))\psi^*(y) = 1$, we have

$$\begin{aligned}
v &= c\psi^*(y) v \\
&= c(T\mathbf{u})^\beta(y) \\
&= (T(c \cdot \mathbf{u}))^\beta(y),
\end{aligned}$$

where the last equality holds because $c \in \mathbb{R}$ and $(T\mathbf{u})^\beta(y) \in F$. Finally,

$$\begin{aligned}
(T(\varphi f))^\beta(y) &= (T\mathbf{w})^\beta(y), \text{ where } w = \varphi(h(y)) f(h(y)) = cu, \\
&= (T(c \cdot \mathbf{u}))^\beta(y) = v.
\end{aligned}$$

Claim 2. There exists (y_α) in Z converging to y_0 such that $\psi^*(y_\alpha) \in \mathbb{R}$ for all α.
Proof of Claim 2. It is enough to show that for all open neighborhoods V of y_0 in βY, there exists $y \in V \cap Z$ with $\psi^*(y) \in \mathbb{R}$. Let V be an open neighborhood of y_0 in βY and choose $y_1 \in V \cap Y$. Since $\psi^*(y_1) = \psi(y_1) \in \mathbb{R}$, there is an open neighborhood V_1 of y_1 in βY such that $\psi^*(y) \in \mathbb{R}$ for all $y \in V_1$. Now $V \cap V_1$ is an open set in βY containing y_1 and thus is nonempty. Since Z is dense in βY, $Z \cap V \cap V_1 \neq \emptyset$. If $y \in Z \cap V \cap V_1$, then $y \in Z \cap V$ and $\psi^*(y) \in \mathbb{R}$.

Returning to the proof of the proposition, let (y_α) be chosen using Claim 2. It follows from Claim 1 that

$$v = (T(\varphi f))^\beta(y_\alpha) \text{ for all } \alpha.$$

Thus, $v = (T\,(\varphi f))^{\beta}\,(y_0)$. On the other hand, by Proposition 2,

$$(T\,(\varphi f))^{\beta}\,(y_0) = (T\mathbf{u}_0)^{\beta}\,(y_0)\,, \text{ where } u_0 = \varphi\,(x_0)\,f\,(x_0) \text{ and } x_0 = h\,(y_0)\,,$$

$$= (T\mathbf{0})^{\beta}\,(y_0)\,, \text{ as } \varphi\,(x_0) = \frac{1}{\psi^*\,(y_0)} = 0,$$

$$= 0.$$

Hence $v = 0$, contrary to the choice of v. $\qquad\qquad\square$

Theorem 10. *Let X and Y be realcompact spaces and let E and F be Hausdorff topological vector spaces. Suppose that $T : C(X, E) \to C(Y, F)$ is a vector space isomorphism that preserves common zeros. Then there are a homeomorphism $h : Y \to X$ and, for each $y \in Y$, a vector space isomorphism $S_y : E \to F$ so that*

(4) $Tf(y) = S_y(f(h(y)))$ *for all $f \in C(X, E)$ and all $y \in Y$.*

Conversely, if a vector space isomorphism $T : C(X, E) \to C(Y, F)$ has the form (4), then T preserves common zeros.

Proof. By Theorem 6, there exist homeomorphisms $h : Z \to X$, $k : W \to Y$ so that $h \cup k^{-1} : Z \cup Y \to X \cup W$ is a homeomorphism. According to Proposition 9, $Z \subseteq \upsilon Y$. Since Y is realcompact, we deduce that $Z \subseteq Y$ and hence h is a restriction of k^{-1}. Similarly, $W \subseteq X$. Then $k^{-1}\,(Y) = W \subseteq X = h\,(Z)\,.$ So we must have $X = W$ and $Y = Z$. Therefore, h is a homeomorphism from Y onto X.

For each $y \in Y$, define a linear operator $S_y : E \to F$ by

$$S_y\,(u) = (T\mathbf{u})\,(y)\,.$$

If $u \in \ker S_y$, then $y \in Z\,(T\mathbf{u})$. Thus $Z\,(T\mathbf{u}) \neq \emptyset$ and hence $Z\,(\mathbf{u}) \neq \emptyset$. Consequently, $u = 0$. The surjectivity of S_y follows easily from that of T. Therefore, each S_y is a linear isomorphism. Since $Z = Y$, it follows from Theorem 6 that

$$(Tf)\,(y) = S_y(f\,(h(y))) \text{ for all } y \in Y.$$

The converse is clear. $\qquad\qquad\square$

The following corollary of Theorem 10 contains the result conjectured in [10]. The original conjecture was solved by Chen, Chen and Wong [8] and independently by Ercan and Önal [11].

Theorem 11. *Let X and Y be realcompact spaces and let E and F be Hausdorff topological vector lattices. Suppose that $T : C(X, E) \to C(Y, F)$ is a vector lattice isomorphism so that*

$$Z(f) \neq \emptyset \iff Z(Tf) \neq \emptyset.$$

Then there is a homeomorphism $h : Y \to X$ and, for each $y \in Y$, a vector lattice isomorphism $S_y : E \to F$ so that

$$Tf(y) = S_y(f\,(h(y))) \text{ for all } f \in C(X, E) \text{ and all } y \in Y.$$

Proof. We first show that T preserves common zeros. Indeed, if $f_1, \cdots, f_k \in C(X, E)$, set $f = |f_1| \vee \cdots \vee |f_k|$. It is clear that $Z(f) = \cap_{i=1}^{k} Z(f_i)$. Since T is a lattice isomorphism, $Tf = |Tf_1| \vee \cdots \vee |Tf_k|$. Thus,

$$\emptyset \neq \cap_{i=1}^{k} Z(f_i) = Z(f)$$
$$\Leftrightarrow Z(Tf) \neq \emptyset, \text{ by hypothesis,}$$
$$\Leftrightarrow \emptyset \neq Z(|Tf_1| \vee \cdots \vee |Tf_k|) = \cap_{i=1}^{k} Z(Tf_i).$$

By Theorem 10 , we obtain a homeomorphism $h : Y \to X$ and vector space isomorphisms S_y, $y \in Y$, satisfying (4). For any $y \in Y$ and $u \in E$, the equation $(T|\mathbf{u}|)(y) = |T\mathbf{u}|(y)$ shows that $S_y(|u|) = |S_y(u)|$. Thus S_y is a vector lattice isomorphism. $\qquad\square$

Theorem 10 holds equally if the scalar field is assumed to be $\mathbb{C}$.

Theorem 12. *Let X and Y be realcompact spaces and let E and F be C^*-algebras. Suppose that $T : C(X, E) \to C(Y, F)$ is a $*$-algebra isomorphism so that*

$$Z(f) \neq \emptyset \iff Z(Tf) \neq \emptyset.$$

Then there is a homeomorphism $h : Y \to X$ and, for each $y \in Y$, a C^*-algebra isomorphism $S_y : E \to F$ so that

$$Tf(y) = S_y(f \circ h(y)) \text{ for all } f \in C(X, E) \text{ and all } y \in Y.$$

Proof. If $f_1, \cdots, f_k \in C(X, E)$, let $f = \sum_{i=1}^{k} f_i f_i^*$. It is clear that $Z(f) = \cap_{i=1}^{k} Z(f_i)$. Since T is a $*$-isomorphism, $Tf = \sum_{i=1}^{k} Tf_i (Tf_i)^*$. Thus,

$$\emptyset \neq \cap_{i=1}^{k} Z(f_i) = Z(f)$$
$$\Leftrightarrow Z(Tf) \neq \emptyset, \text{ by hypothesis,}$$
$$\Leftrightarrow \emptyset \neq Z(\sum_{i=1}^{k} Tf_i (Tf_i)^*) = \cap_{i=1}^{k} Z(Tf_i).$$

Therefore T preserves common zeros. The rest of the proof follows along the lines of the proof of Theorem 11. $\qquad\square$

4. SPACES OF DIFFERENTIABLE FUNCTIONS

In this section, we fix $p, q \in \mathbb{N}$ and let X and Y be open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. The results of §2 are applied to spaces of differentiable functions $C^m(X, E)$ and $C^n(Y, F)$, where $m, n \in \mathbb{N} \cup \{0, \infty\}$. Here $C^m(X, E)$ denotes the space of functions from X into E having continuous partial derivatives of all order $< m + 1 (\infty + 1 = \infty)$. Note that the spaces $C^m(X)$ and $C^n(Y)$ are almost normal and thus $C^m(X, E)$ and $C^n(Y, F)$ are almost normally multiplicative.

Lemma 13. *Let X and Y be open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, $p, q \in \mathbb{N}$. Suppose that Z is a dense subspace of βY that is homeomorphic to X. Then $Z \subseteq Y$.*

Proof. Suppose that $h : Z \to X$ is a homeomorphism. If $Z \nsubseteq Y$, then there exists $y_0 \in Z \smallsetminus Y$ such that $h(y_0) = x_0 \in X$. For each n, let U_n denote the open set $h^{-1}\left(B\left(x_0, \frac{1}{n}\right)\right)$ in Z. Then $U_n = V_n \cap Z$ for some open subset V_n of βY. Since Y is locally compact, it is open in βY. Thus, $V_n \cap Y$ is a nonempty open set in βY. Therefore, $V_n \cap Y \cap Z \neq \emptyset$. For each n, pick $y_n \in V_n \cap Y \cap Z$. Since $h(y_n) \to x_0$ and h^{-1} is continuous, (y_n) converges to y_0. We may assume without loss of generality that (y_n) is pairwise distinct and has no accumulation point in Y. There exists $g \in C(Y)$ such that $0 \leq g \leq 1$, $g(y_{2n-1}) = 0$ and $g(y_{2n}) = 1$ for all n. Consider the continuous extension $g^{\#} : \beta Y \to [0,1]$ of g. By the continuity of $g^{\#}$,

$$1 = \lim_{n \to \infty} g(y_{2n}) = g^{\#}(y_0) = \lim_{n \to \infty} g(y_{2n-1}) = 0,$$

a contradiction. $\qquad\square$

Lemma 14. *Let φ be a real-valued function on Y and f be a function in $C(Y, F)$ that is never zero. Assume that $\lim_{y \to y_0} \varphi(y) f(y) = v$ exists. Then $\lim_{y \to y_0} \varphi(y) = a$ exists and $a \cdot f(y_0) = v$.*

Proof. We first show that φ is bounded in a neighborhood of y_0. Suppose otherwise. Then there is a sequence (y_n) converging nontrivially to y_0 such that $|\varphi(y_n)| \geq n$ for all n. Since $\lim_{y \to y_0} \varphi(y) f(y) = v$ exists, for any circled neighborhood U of 0, $\varphi(y_n) f(y_n) \in v + U$ for sufficiently large n. Thus

$$f(y_n) - \frac{v}{\varphi(y_n)} \in \frac{1}{\varphi(y_n)} U \subseteq U$$

for sufficiently large n. Hence $\lim_{n \to \infty}\left(f(y_n) - \frac{v}{\varphi(y_n)}\right) = 0$. By the continuity of f, $f(y_0) = 0$, a contradiction.

Since φ is bounded in a neighborhood of y_0, every sequence (y_n) that converges nontrivially to y_0 has a subsequence (y_{n_k}) such that $(\varphi(y_{n_k}))$ converges. Suppose that (y_n) and (z_n) are sequences converging nontrivially to y_0 such that

$$\lim_{n \to \infty} \varphi(y_n) = L_1 \text{ and } \lim_{n \to \infty} \varphi(z_n) = L_2.$$

Then

$$L_1 f(y_0) = \lim_{n \to \infty} \varphi(y_n) f(y_n) = \lim_{n \to \infty} \varphi(z_n) f(z_n) = L_2 f(y_0).$$

Since $f(y_0) \neq 0$, $L_1 = L_2$. Hence $\lim_{y \to y_0} \varphi(y) = a$ exists. Clearly $a \cdot f(y_0) = v$. $\qquad\square$

Let $r \in \mathbb{N}$. A *multi-index* λ is an r-tuple $(\lambda_1, ..., \lambda_r)$ with entries in $\mathbb{N} \cup \{0\}$, which will also be regarded as a vector in $\mathbb{R}^r$. The *order* of λ is $|\lambda| = \lambda_1 + \cdots + \lambda_r$. If f is a function of r variables, we denote by $\partial^{\lambda} f$ the partial derivative $\left(\partial^1\right)^{\lambda_1} \cdots \left(\partial^r\right)^{\lambda_r} f$.

Lemma 15. *Let $n \in \mathbb{N} \cup \{0\}$. If φ is real-valued function on Y and $\varphi f \in C^n(Y, F)$ for some $f \in C^n(Y, F)$ that is never zero, then $\varphi \in C^n(Y)$.*

Proof. The case $n = 0$ follows easily from Lemma 14. We prove the remaining cases by induction. Assume that $n = 1$. Let λ be a multi-index with $|\lambda| = 1$. For all $y_0 \in Y$ and all $t \neq 0$, set $y_t = y_0 + t\lambda$. Then

$$\lim_{t \to 0} \frac{\varphi(y_t) - \varphi(y_0)}{t} f(y_t)$$
$$= \lim_{t \to 0} \left\{ \frac{\varphi(y_t) f(y_t) - \varphi(y_0) f(y_0)}{t} - \varphi(y_0) \frac{f(y_t) - f(y_0)}{t} \right\}$$

exists and is equal to

$$\partial^\lambda(\varphi f)(y_0) - \varphi(y_0) \partial^\lambda f(y_0).$$

By Lemma 14, $\partial^\lambda \varphi(y_0) = \lim_{t \to 0} \frac{\varphi(y_t) - \varphi(y_0)}{t}$ exists and

$$(5) \qquad \partial^\lambda \varphi(y_0) \cdot f(y_0) = \partial^\lambda(\varphi f)(y_0) - \varphi(y_0) \partial^\lambda f(y_0).$$

From the case $n = 0$, we know that φ is continuous on Y. Together with the assumptions that $\partial^\lambda(\varphi f)$ and $\partial^\lambda f$ are continuous and that f is never zero, we can deduce using (5) and Lemma 14 that $\partial^\lambda \varphi$ is continuous. Since this is true for all multi-indices λ with $|\lambda| = 1$, we conclude that $\varphi \in C^1(X)$.

Suppose that the lemma is true for some integer $n \geq 1$. Assume that $f, \varphi f \in C^{n+1}(Y, F)$, with f never zero on Y. By the inductive hypothesis, $\varphi \in C^n(Y)$. Also, for any multi-index λ with $|\lambda| = 1$, we have by (5)

$$\left(\partial^\lambda \varphi \right) \cdot f = \partial^\lambda(\varphi f) - \varphi \partial^\lambda f.$$

In particular, $\partial^\lambda \varphi \cdot f \in C^n(Y, F)$. By the inductive hypothesis, $\partial^\lambda \varphi \in C^n(Y)$. Hence $\varphi \in C^{n+1}(Y)$. $\qquad \square$

Theorem 16. *Let X and Y be open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, $p, q \in \mathbb{N}$, and let E and F be Hausdorff topological vector spaces. Suppose that $m, n \in \mathbb{N} \cup \{0, \infty\}$ and $T : C^m(X, E) \to C^n(Y, F)$ is a vector space isomorphism so that T preserves common zeros. Then $p = q$ and $m = n$. Moreover, there are a C^n-diffeomorphism $h : Y \to X$ and, for each $y \in Y$, a vector space isomorphism $S_y : E \to F$ so that*

$$(6) \qquad Tf(y) = S_y(f \circ h(y)) \text{ for all } f \in C^n(X, E) \text{ and all } y \in Y.$$

Conversely, if a vector space isomorphism $T : C^m(X, E) \to C^n(Y, F)$ has the form (6), then T preserves common zeros.

Proof. Applying Theorem 6 with $A(X, E) = C^m(X, E)$ and $A(Y, F) = C^n(Y, F)$, there exist homeomorphisms $h : Z \to X$, $k : W \to Y$ so that $h \cup k^{-1} : Z \cup Y \to X \cup W$ is a homeomorphism. According to Lemma 13, $Z \subseteq Y$ and $W \subseteq X$. Following the arguments as in the proof of Theorem 10, $h : Y \to X$ is a homeomorphism. Since X and Y, which are open subsets of open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, are homeomorphic, it follows from the Brouwer Domain Invariance Theorem [9, Chapter XVII, Theorem 3.1] that $p = q$.

We now show that $h \in C^n (Y, X)$. Let $v \in E \smallsetminus \{0\}$ be fixed and set $f_i \in C^m (X, E)$, $i = 1, \cdots, p$, to be the function $f_i (x) = x_i v$ if $x = (x_1, \cdots, x_p) \in X$. By Proposition 2, for all $y \in Y$,

$$\begin{aligned}
(T f_i) (y) = T \mathbf{u}_i (y), \ & \text{where } u_i = f_i (h (y)) \\
& = (T (h_i (y) \mathbf{v})) (y), \ \text{where } h (y) = (h_1 (y), \cdots, h_p (y)), \\
& = h_i (y) T \mathbf{v} (y).
\end{aligned}$$

Since T preserves common zeros, $T\mathbf{v}$ is never zero. Applying Lemma 15 to the real-valued functions h_i yields that $h_i \in C^n (Y)$. Hence $h = (h_1, \cdots, h_p) \in C^n (Y, X)$. Similarly, $h^{-1} \in C^m (X, Y)$.

Next, we show that $m = n$. Given $\psi \in C^m (Y)$, $\varphi = \psi \circ h^{-1} \in C^m (X)$. Fix $u \in E \smallsetminus \{0\}$, and let $f = \varphi \cdot \mathbf{u}$. Then $f \in C^m (X, E)$ and hence $Tf \in C^n (Y, F)$. By Proposition 2,

$$\begin{aligned}
(Tf) (y) = T \mathbf{w} (y), \ & \text{where } w = f (h (y)) = \varphi (h (y)) u, \\
& = T (\varphi (h (y)) \mathbf{u}) (y) \\
& = T (\psi (y) \mathbf{u}) (y) \\
& = \psi (y) T \mathbf{u} (y).
\end{aligned}$$

Since $T\mathbf{u}, Tf \in C^n (Y, F)$ and $T\mathbf{u}$ is never 0 on Y, we conclude from Lemma 15 that $\psi \in C^n (Y)$. Hence $C^m (Y) \subseteq C^n (Y)$ and thus $m \geq n$. By symmetry, $m \leq n$.

For each $y \in Y$, define $S_y : E \to F$ by

$$S_y (u) = (T\mathbf{u}) (y).$$

From the proof of Theorem 10, we see that S_y is a vector space isomorphism that satisfies (6). The converse is clear. $\square$

5. Automatic continuity

In this section, we investigate the continuity properties of linear isomorphic mappings between spaces of differentiable functions that preserves common zeros. If $T : C^m (X, E) \to C^n (Y, F)$ is a linear isomorphism that preserves common zeros, where X and Y are open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, we have by Theorem 16 that $p = q$, $m = n$. Also, there are a C^m-diffeomorphism $h : X \to Y$ and vector space isomorphisms $S_y : E \to F, y \in Y$, satisfying (6). Define $J : C^m (Y, F) \to C^m (X, F)$ by $(Jg) (x) = g (h^{-1} (x))$. Clearly $JT : C^m (X, E) \to C^m (X, F)$ is a vector space isomorphism preserving common zeros and $(JTf) (x) = S_x (f (x))$ for all $f \in C^m (Y, F)$ and all $x \in X$. Therefore, in considering the continuity of T and the associated map $\Phi : Y \times E \to F$, $\Phi (y, u) = S_y (u)$, there is no loss of generality in assuming that $X = Y$ and that h is the identity map.

Proposition 17. *Let X be an open subset of $\mathbb{R}^p$, $p \in \mathbb{N}$, E and F be Hausdorff topological vector spaces and $m \in \mathbb{N} \cup \{\infty\}$. Assume that $\Phi : X \times E \to F$ satisfies*

(1) *For all $u \in E$, $\Phi(\cdot, u)$ belongs to $C^m(X, F)$;*
(2) *For all $x \in X$, $\Phi(x, \cdot)$ is a linear operator from E into F;*
(3) Φ *is sequentially continuous.*

For all $f \in C^m(X, E)$, define $Tf(x) = \Phi(x, f(x))$. Then $Tf \in C^m(X, F)$ and T is a linear operator from $C^m(X, E)$ to $C^m(X, F)$.

Proof. The proposition holds for $m = \infty$ if it holds for all $m \in \mathbb{N}$. For a fixed $m \in \mathbb{N}$ the proposition is a special case of the following claim:

Claim. If $f \in C^m(X, E)$ and $|\lambda| \leq m$, then $\theta_\lambda(x) = \Phi(x, \partial^\lambda f(x))$ belongs to $C^{m-|\lambda|}(X, F)$.

We prove the claim by induction on $m - |\lambda|$. Suppose that $m - |\lambda| = 0$. If $f \in C^m(X, E)$, then $\partial^\lambda f \in C(X, E)$. Thus $\partial^\lambda f(x_n) \to \partial^\lambda f(x_0)$ whenever $x_n \to x_0$. It follows from the sequential continuity of Φ that $\theta_\lambda(x_n) = \Phi(x_n, \partial^\lambda f(x_n)) \to \Phi(x_0, \partial^\lambda f(x_0)) = \theta_\lambda(x_0)$. Hence θ_λ is sequentially continuous on the metric space X. Therefore, $\theta_\lambda \in C(X, E)$.

Suppose that the claim is true for $m - |\lambda| = k$. Let m and λ be such that $m - |\lambda| = k + 1$. If $1 \leq i \leq p$, let e_i denote the ith coordinate unit vector of $\mathbb{R}^p$. For $t \neq 0$ and $x_0 \in X$,

$$\frac{\theta_\lambda(x_0 + te_i) - \theta_\lambda(x_0)}{t} = \frac{\Phi\left(x_0 + te_i, \partial^\lambda f(x_0 + te_i)\right) - \Phi\left(x_0, \partial^\lambda f(x_0)\right)}{t}$$

$$= \frac{\Phi\left(x_0 + te_i, \partial^\lambda f(x_0 + te_i)\right) - \Phi\left(x_0 + te_i, \partial^\lambda f(x_0)\right)}{t}$$

$$+ \frac{\Phi\left(x_0 + te_i, \partial^\lambda f(x_0)\right) - \Phi\left(x_0, \partial^\lambda f(x_0)\right)}{t}$$

$$= \Phi\left(x_0 + te_i, \frac{\partial^\lambda f(x_0 + te_i) - \partial^\lambda f(x_0)}{t}\right) + \frac{g(x_0 + te_i) - g(x_0)}{t},$$

where $g(\cdot) = \Phi(\cdot, \partial^\lambda f(x_0)) \in C^m(X, F)$, according to (1). If $t_n \neq 0$, $t_n \to 0$, then by (3),

$$\lim_{n \to \infty} \frac{\theta_\lambda(x_0 + t_n e_i) - \theta_\lambda(x_0)}{t_n} = \Phi\left(x_0, \partial^i \partial^\lambda f(x_0)\right) + \partial^i g(x_0).$$

Hence

$$\partial^i \theta_\lambda(x_0) = \theta_{\lambda + e_i}(x_0) + \partial^i g(x_0).$$

Since $m - |\lambda + e_i| = m - |\lambda| - 1 = k$, $\theta_{\lambda + e_i} \in C^k(X, F)$ by induction. Also, $\partial^i g \in C^{m-1}(X, F) \subseteq C^k(X, F)$. It follows that $\partial^i \theta_\lambda \in C^k(X, F)$ for all $1 \leq i \leq p$. Hence $\theta_\lambda \in C^{k+1}(X, F) = C^{m-|\lambda|}(X, F)$. □

Let $|x|$ denote the Euclidean norm of a vector $x \in \mathbb{R}^p$. A C^∞-function $\varphi : \mathbb{R}^p \to \mathbb{R}$ is called a C^∞-*bump* if $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^p$, $\varphi(x) = 1$ if $|x| \leq \frac{1}{2}$ and $\varphi(x) = 0$ if $|x| \geq 1$.

Proposition 18. *Let E be a Hausdorff topological vector space. Suppose that φ is a C^∞-bump on $\mathbb{R}^p$, (u_n) is a bounded sequence in E and (x_n) is*

a sequence in $\mathbb{R}^p$ such that $|x_{n+1} - x_0| < \frac{1}{3}|x_n - x_0| < 1$ for all n. Set $\varphi_n(x) = \varphi\left(\frac{x - x_n}{r_n/2}\right)$, where $r_n = |x_n - x_0|$. If $m \in \mathbb{N}$ and (c_n) is a sequence of real numbers such that

$$\lim_{n \to \infty} \frac{c_n}{r_n^m} = 0,$$

then $f = \sum c_n \varphi_n \mathbf{u}_n \in C^m(\mathbb{R}^p, E)$.

Proof. For a given n and any a, b such that

$$\frac{3r_{n+1}}{2} < a < \frac{r_n}{2} \text{ and } \frac{3r_n}{2} < b < \frac{r_{n-1}}{2} \ (r_0 = \infty),$$

set $A_{a,b} = \{x \in \mathbb{R}^p : a < |x - x_0| < b\}$. Since for all k, φ_k is supported on a ball centered at x_k with radius $r_k/2$, we see that $f = c_n \varphi_n \mathbf{u}_n$ on $A_{a,b}$. Therefore, f is infinitely differentiable on $A_{a,b}$ and

$$\partial^\lambda f = c_n \partial^\lambda \varphi_n \cdot \mathbf{u}_n \text{ on } A_{a,b}, \ |\lambda| \leq m.$$

It follows easily that f is infinitely differentiable on $\mathbb{R}^p \setminus \{x_0\}$ and

$$(7) \qquad \partial^\lambda f = \sum_n c_n \partial^\lambda \varphi_n \cdot \mathbf{u}_n \text{ on } \mathbb{R}^p \setminus \{x_0\}, \ |\lambda| \leq m.$$

Claim. If $|\lambda| \leq m$, then $\partial^\lambda f(x_0) = 0$.

Indeed, if $|\lambda| = 0$, $f(x_0) = \sum c_n \varphi_n(x_0) u_n = 0$. Suppose that the claim holds for all λ with $|\lambda| = k$ for some fixed $k < m$. Given any λ, $|\lambda| = k - 1$, $1 \leq i \leq p$ and $t \neq 0$,

$$\frac{\partial^\lambda f(x_0 + te_i) - \partial^\lambda f(x_0)}{t} = \frac{\partial^\lambda f(x_0 + te_i)}{t}$$

by the inductive hypothesis. By (7),

$$\partial^\lambda f(x_0 + te_i) = \begin{cases} c_n \partial^\lambda \varphi_n(x_0 + te_i) u_n & \text{if } \frac{r_n}{2} < |t| < \frac{3r_n}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Let U be an open neighborhood of zero in E. Since (u_n) is a bounded sequence, there exists $\varepsilon > 0$ such that $(\alpha u_n) \subseteq U$ for all $|\alpha| < \varepsilon$. Choose N such that $\frac{c_n}{(r_n/2)^k}\|\partial^\lambda \varphi\|_\infty < \varepsilon$ for all $n > N$. Suppose that $0 < |t| < \frac{3r_N}{2}$. Then

$$\frac{\partial^\lambda f(x_0 + te_i)}{t} = \begin{cases} \frac{c_n}{t} \partial^\lambda \varphi_n(x_0 + te_i) u_n & \text{if } \frac{r_n}{2} < |t| < \frac{3r_n}{2}, n > N, \\ 0 & \text{otherwise.} \end{cases}$$

But when $\frac{r_n}{2} < |t| < \frac{3r_n}{2}$ for some $n > N$,

$$\left|\frac{c_n \partial^\lambda \varphi_n(x_0 + te_i)}{t}\right| \leq \frac{c_n}{r_n/2} \frac{\|\partial^\lambda \varphi\|_\infty}{(r_n/2)^{|\lambda|}}$$

$$= \frac{c_n}{(r_n/2)^k}\|\partial^\lambda \varphi\|_\infty < \varepsilon.$$

Therefore,

$$\frac{\partial^\lambda f\left(x_0 + te_i\right) - \partial^\lambda f\left(x_0\right)}{t} \in U \text{ if } 0 < |t| < \frac{3r_N}{2}.$$

This shows that $\partial^i \partial^\lambda f\left(x_0\right) = 0$. So the claim is verified by induction.

For $|\lambda| = m$ and $x \in \mathbb{R}^p$,

$$\partial^\lambda f\left(x\right) = \begin{cases} c_n \partial^\lambda \varphi_n\left(x\right) u_n & = & c_n \left(\frac{2}{r_n}\right)^{|\lambda|} \partial^\lambda \varphi\left(\frac{x - x_n}{r_n/2}\right) u_n \\ & & \text{if } \frac{r_n}{2} < |x - x_n| < \frac{3r_n}{2}, \\ 0 & & \text{otherwise.} \end{cases}$$

Since (u_n) is bounded and $\lim\limits_{n\to\infty} c_n \left(\frac{2}{r_n}\right)^{|\lambda|} \left\|\partial^\lambda \varphi\right\|_\infty = 0$, $\lim\limits_{x\to x_0} \partial^\lambda f\left(x\right) = 0$. Hence $\partial^\lambda f$ is continuous at x_0. Thus $f \in C^m\left(\mathbb{R}^p, E\right)$. $\hfill\square$

Together with Proposition 17, the next result characterizes when a map T, defined in terms of the associated map Φ, sends functions from $C^m\left(X, E\right)$ to $C^m\left(X, F\right)$, in the case where E and F are locally convex and E is metrizable.

Theorem 19. *Let X be an open subset of $\mathbb{R}^p$, $p \in \mathbb{N}$. Suppose that E and F are Hausdorff topological vector spaces and that F is locally convex. If, for some $m \in \mathbb{N}$, the map $\Phi : X \times E \to F$ has the property that $Tf(x) = \Phi(x, f(x))$ defines a linear operator T from $C^m(X, E)$ to $C^m(X, F)$. Then Φ has the following properties.*

    (1) For all $u \in E$, $\Phi(\cdot, u)$ belongs to $C^m(X, F)$;
    (2) For all $x \in X$, $\Phi(x, \cdot)$ is a linear operator from E into F;
    (3′) If (x_n) is a sequence in X converging to some $x_0 \in X$ and (u_n) is a bounded sequence in E, then $(\Phi(x_n, u_n))$ is a bounded sequence in F.

Moreover, if E is locally convex metrizable, then Φ is continuous.

Proof. We first show that the "moreover" statement follows from (1), (2), and (3′). Suppose that E is locally convex metrizable and Φ is not continuous. Then there exist sequences (x_n) and (u_n) converging to x_0 and u_0 in X and E respectively such that $\Phi\left(x_n, u_n\right) \nrightarrow \Phi\left(x_0, u_0\right)$. It follows from the local convexity of F that there is a continuous seminorm ρ on F such that

$$d_n = \rho\left(\Phi\left(x_n, u_n\right), \Phi\left(x_0, u_0\right)\right) \nrightarrow 0.$$

Let $v_n = u_n - u_0$. Then (v_n) converges to 0. Note that

$$\Phi\left(x_n, v_n\right) - \Phi\left(x_0, 0\right) = \Phi\left(x_n, u_n\right) - \Phi\left(x_n, u_0\right) - \Phi\left(x_0, 0\right)$$
$$= \left(\Phi\left(x_n, u_n\right) - \Phi\left(x_0, u_0\right)\right) + \left(\Phi\left(x_0, u_0\right) - \Phi\left(x_n, u_0\right)\right).$$

By (1), $\lim\limits_{n\to\infty}\left(\Phi\left(x_0, u_0\right) - \Phi\left(x_n, u_0\right)\right) = 0$. Therefore, $\Phi\left(x_n, v_n\right) \nrightarrow \Phi\left(x_0, 0\right)$. This shows that we may assume $u_0 = 0$ without loss of generality.

By using a subsequence, we may further assume that for some $c > 0$, $d_n = \rho\left(\Phi\left(x_n, u_n\right)\right) \geq c$ for all n. Since E is locally convex metrizable, there exists a sequence of continuous seminorms (ρ_k) on E which determines the topology on E. For all k, $\lim\limits_{n\to\infty} \rho_k\left(u_n\right) = 0$. Thus there exists a sequence (α_n) of positive numbers diverging to ∞ such that $\lim\limits_{n\to\infty} \alpha_n\rho_k\left(u_n\right) = 0$ for all k. Now $(\alpha_n u_n)$ is a bounded sequence in E. But $\rho\left(\Phi\left(x_n, \alpha_n u_n\right)\right) = \alpha_n d_n \geq \alpha_n c \to \infty$. Hence $(3')$ fails.

We now turn to proving that Φ satisfies (1), (2) and $(3')$. The first two parts are clear. Suppose that $(3')$ fails. Then there exist sequences (x_n) and (u_n) in X and E respectively, with (x_n) converging to some $x_0 \in X$, (u_n) bounded in E, such that $(\Phi(x_n, u_n))$ is unbounded in F. Take a continuous seminorm ρ on F such that $(\rho(\Phi(x_n, u_n)))$ is unbounded. Using a subsequence if necessary, we may assume that

$$3r_{n+1} = 3\left|x_{n+1} - x_0\right| < \left|x_n - x_0\right| = r_n < 1 \text{ for all } n$$

and that $\lim\limits_{n\to\infty} \rho(\Phi(x_n, u_n)) = \infty$. Set $c_n = \frac{1}{\rho(\Phi(x_n, u_n))} r_n^m$ for all n. By Proposition 18, $f = \sum c_n \varphi_n \mathbf{u}_n \in C^m\left(X, E\right)$. Therefore, $g(x) = \Phi(x, f(x)) \in C^m\left(X, F\right)$. By (2), $g(x) = 0$ whenever $f(x) = 0$. For any n, a, and b such that

$$\frac{3r_{n+1}}{2} < a < \frac{r_n}{2} \text{ and } \frac{3r_n}{2} < b < \frac{r_{n-1}}{2},$$

$f = 0$ on $A_{a,b} = \{x \in X : a < |x - x_0| < b\}$. Thus $g = 0$ on $A_{a,b}$. By continuity, we have $\partial^\lambda g\left(x_0\right) = 0$ if $|\lambda| \leq m$. Let $i : F \to \overline{F_\rho}$ be the natural map, where F_ρ is the quotient space $F/\rho^{-1}\{0\}$ normed by ρ and $\overline{F_\rho}$ is its completion. Then $G = i \circ g \in C^m(X, \overline{F_\rho})$. By Taylor's Formula (see e.g., [18, p.115]), denoting by $(x_n - x_0)^{(m)}$ the vector in $(\mathbb{R}^p)^m$ with coordinates $x_n - x_0$ repeated m-times,

$$G\left(x_n\right) = G\left(x_0\right) + \sum_{k=1}^m \frac{D^{(k)}G\left(x_0\right)\left(x_n - x_0\right)^{(m)}}{k!} + E_n,$$

where $\lim\limits_{n\to\infty} \frac{E_n}{|x_n-x_0|^m} = 0$. Since $D^{(k)}g\left(x_0\right) = 0$ for $0 \leq k \leq m$, we have $D^{(k)}G\left(x_0\right) = 0$ for $0 \leq k \leq m$. Thus $\lim\limits_{n\to\infty} \frac{G(x_n)}{|x_n-x_0|^m} = 0$. It follows that

$$0 = \lim_{n\to\infty} \frac{\rho\left(g\left(x_n\right)\right)}{\left|x_n - x_0\right|^m} = \lim_{n\to\infty} \frac{\rho\left(\Phi(x_n, f(x_n))\right)}{r_n^m}$$
$$= \lim_{n\to\infty} \frac{\rho\left(\Phi(x_n, c_n u_n)\right)}{r_n^m} = \lim_{n\to\infty} \frac{c_n \rho\left(\Phi(x_n, u_n)\right)}{r_n^m} = 1,$$

a contradiction. $\qquad\qquad\square$

When E and F are locally convex Fréchet spaces, Theorem 19 can be strengthened to yield continuity of the partial derivatives of Φ with respect to the coordinates of x. The results also holds for $m = \infty$. The assumption

of completeness is crucial here as Examples 24 and 27 will show. The idea for Theorem 20 comes from [3], especially §5.

Theorem 20. *Let X be an open subset of $\mathbb{R}^p$, $p \in \mathbb{N}$, and let $m \in \mathbb{N} \cup \{\infty\}$. Assume that E and F are locally convex Fréchet spaces. Suppose that the map $\Phi : X \times E \to F$ has the property that $Tf(x) = \Phi(x, f(x))$ defines a linear operator T from $C^m(X, E)$ into $C^m(X, F)$.*

(1) *If $C^m(X, E)$ and $C^m(X, F)$ are endowed with complete linear metric topologies that are stronger than the respective topologies of pointwise convergence, then T is continuous;*
(2) *For any λ with $|\lambda| < m + 1$, the map $\Phi_\lambda : X \times E \to F$ defined by $\Phi_\lambda(x, u) = (\partial^\lambda T\mathbf{u})(x)$ is continuous. (We adopt the convention $\infty + 1 = \infty$).*

In the next two lemmas, E and F are locally convex Fréchet spaces.

Lemma 21. *Assume that Φ and T satisfy the hypotheses of Theorem 20 for some $m \in \mathbb{N} \cup \{\infty\}$. Let (u_n) be a bounded sequence in E and (x_n) be a sequence in $\mathbb{R}^p$ converging to a point x_0 such that*

$$3r_{n+1} < r_n < 1 \text{ for all } n,$$

where $r_n = |x_n - x_0|$. If (c_n) is a sequence of real numbers such that

$$\lim_{n \to \infty} \frac{c_n}{r_n^m} = 0 \text{ if } m \in \mathbb{N} ,$$

$$\lim_{n \to \infty} \frac{c_n}{r_n^k} = 0 \text{ for all } k \in \mathbb{N}, \text{ if } m = \infty,$$

then

$$\lim_{n \to \infty} \frac{c_n \Phi(x_n, u_n)}{r_n^m} = 0 \text{ if } m \in \mathbb{N} ,$$

$$\lim_{n \to \infty} \frac{c_n \Phi(x_n, u_n)}{r_n^k} = 0 \text{ for all } k \in \mathbb{N}, \text{ if } m = \infty.$$

Proof. If $m \in \mathbb{N}$, the conclusion follows since $(\Phi(x_n, u_n))$ is a bounded sequence by $(3')$ of Theorem 19. Consider the case $m = \infty$. Let φ be a C^∞-bump on $\mathbb{R}^p$ and set $\varphi_n(x) = \varphi\left(\frac{x - x_n}{r_n/2}\right)$. According to Proposition 18, $f = \sum c_n \varphi_n \mathbf{u}_n \in C^\infty(X, E)$. Let $g(x) = Tf(x) = \Phi(x, f(x))$. Then $g \in C^\infty(X, F)$. Using the same proof as in Theorem 19, we find that for any continuous seminorm ρ on F and any $k \in \mathbb{N}$,

$$0 = \lim_{n \to \infty} \frac{\rho(g(x_n))}{|x_n - x_0|^k} = \lim_{n \to \infty} \frac{\rho(\Phi(x_n, f(x_n)))}{r_n^k}$$

$$= \lim_{n \to \infty} \frac{\rho(\Phi(x_n, c_n u_n))}{r_n^k} = \lim_{n \to \infty} \rho\left(\frac{c_n \Phi(x_n, u_n)}{r_n^k}\right).$$

The lemma follows since the topology of F is determined by continuous seminorms. □

Lemma 22. *Assume that Φ and T satisfy the hypotheses of Theorem 20 for some $m \in \mathbb{N} \cup \{\infty\}$. Given any compact set $K \subseteq X$, there are only countably many $x \in K$ at which $\Phi(x, \cdot) : E \to F$ is not continuous.*

Proof. Let ρ be a continuous seminorm on F and $q : F \to F_\rho$ be the quotient map, where F_ρ is the quotient space $F/\rho^{-1}\{0\}$ normed by ρ. We claim that $q \circ \Phi(x, \cdot) : E \to F_\rho$ is continuous for all but finitely many $x \in K$. Suppose, to the contrary, that there exists a sequence (x_n) converging to x_0 in K such that $q \circ \Phi(x_n, \cdot)$ is discontinuous for all $n \in \mathbb{N}$. By taking a subsequence if necessary, we may assume that $3r_{n+1} < r_n < 1$ for all n, where $r_n = |x_n - x_0|$. Let $c_n = r_n^n$ if $m = \infty$ and $c_n = r_n^m$ if $m \in \mathbb{N}$. Suppose that the topology on E is determined by a sequence of seminorms (ρ_k). For every n, by the discontinuity of $q \circ \Phi(x_n, \cdot)$, there exists $u_n \in E$ such that $\rho_k(u_n) \leq 1, 1 \leq k \leq n$, and

$$\rho(\Phi(x_n, u_n)) \geq \frac{1}{c_n}.$$

The sequence (u_n) is bounded in E. However,

$$\lim_{n \to \infty} \frac{c_n \rho(\Phi(x_n, u_n))}{r_n^k} \neq 0$$

for any $k \in \mathbb{N}$, contrary to Lemma 21. This proves the claim. Since the topology of F is determined by countably many seminorms, the lemma follows. $\qquad\square$

Proof of Theorem 20. (1) It suffices to show that the graph of T is closed. Suppose that $f_n \to f$ in $C^m(X, E)$ and $Tf_n \to g$ in $C^m(X, F)$. Since the topologies on $C^m(X, E)$ and $C^m(X, F)$ are stronger than the topologies of pointwise convergence, $f_n(x) \to f(x)$ and $Tf_n(x) \to g(x)$ for all $x \in X$. At any x where $\Phi(x, \cdot)$ is continuous,

$$Tf_n(x) = \Phi(x, f_n(x)) \to \Phi(x, f(x)) = Tf(x).$$

By Lemma 22, $Tf_n(x) \to Tf(x)$ for all x in a co-countable subset of X. Hence $Tf = g$ on a dense subset of X. By continuity, $Tf = g$.

(2) Endow $C^m(X, E)$ and $C^m(X, F)$ with the respective topologies of uniform convergence on compact sets of all partial derivatives of order $< m + 1$. Then the hypothesis of (1) is satisfied.

If (x_n) and (u_n) are sequences in X and E converging to $x_0 \in X$ and $u_0 \in E$ respectively, then the sequence of constant functions $(\mathbf{u}_n)$ converges to $\mathbf{u}_0$ in the given topology of $C^m(X, E)$. By continuity of T, $(T\mathbf{u}_n)$ converges to $T\mathbf{u}_0$ in $C^m(Y, F)$. Since $K = \{x_n\}_{n=1}^\infty \cup \{x_0\}$ is compact, $\partial^\lambda T\mathbf{u}_n \to \partial^\lambda T\mathbf{u}_0$ uniformly on K for all λ, $|\lambda| < m + 1$. Thus

$\lim_{n\to\infty} \left(\left(\partial^\lambda T\mathbf{u}_n \right)(x_n) - \left(\partial^\lambda T\mathbf{u}_0 \right)(x_n) \right) = 0$. Hence

$$\Phi_\lambda \left(x_n, u_n \right) - \Phi_\lambda \left(x_0, u_0 \right) = \left(\partial^\lambda T\mathbf{u}_n \right)(x_n) - \left(\partial^\lambda T\mathbf{u}_0 \right)(x_0)$$
$$= \left(\partial^\lambda T\mathbf{u}_n \right)(x_n) - \left(\partial^\lambda T\mathbf{u}_0 \right)(x_n)$$
$$+ \left(\partial^\lambda T\mathbf{u}_0 \right)(x_n) - \left(\partial^\lambda T\mathbf{u}_0 \right)(x_0)$$

converges to 0, keeping in mind the continuity of $\partial^\lambda T\mathbf{u}_0$. $\square$

For ease of reference, let us summarize the results from the preceding section and the present one into a unified statement.

Theorem 23. *Let X and Y be open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, $p, q \in \mathbb{N}$, and let E and F be Hausdorff topological vector spaces. Assume that $m, n \in \mathbb{N} \cup \{0, \infty\}$ and $T : C^m(X, E) \to C^n(Y, F)$ is a vector space isomorphism so that T preserves common zeros. Then $p = q$, $m = n$, and there are a C^m-diffeomorphism $h : Y \to X$ and a map $\Phi : Y \times E \to F$ such that*

(a) *For all $u \in E$, $\Phi(\cdot, u)$ belongs to $C^m(Y, F)$;*
(b) *For all $y \in Y$, $\Phi(y, \cdot)$ is a vector space isomorphism from E onto F;*
(c) *$(Tf)(y) = \Phi(y, f(h(y)))$ for all $f \in C^m(X, E)$ and all $y \in Y$;*
(d) *If E and F are locally convex metrizable and $m \in \mathbb{N}$, then Φ is continuous on $Y \times F$. In particular, E and F are isomorphic as topological vector spaces;*
(e) *If E and F and both locally convex Fréchet, then the conclusion in (d) also holds for $m = \infty$. Furthermore, for any λ with $|\lambda| < m + 1$, the map $\Phi_\lambda : Y \times E \to F$ defined by $\Phi_\lambda(y, u) = \left(\partial^\lambda T\mathbf{u} \right)(y)$ is continuous, and T is continuous whenever $C^m(X, E)$ and $C^m(Y, F)$ are given complete linear metric topologies stronger than the respective topologies of pointwise convergence.*

Example 24 demonstrates that neither the continuity of T nor that of Φ_λ, $|\lambda| > 0$, is guaranteed without the completeness of E and F. Similarly, Example 27 shows that the the continuity of Φ itself is not guaranteed if $m = \infty$ and E and F are not assumed to be complete. Denote by c_{00} the space of all finitely supported real sequences endowed with the sup-norm.

Example 24. *For any $m \in \mathbb{N}$, there is a map $\Phi : \mathbb{R} \times c_{00} \to c_{00}$ such that the induced map T given by $Tf(x) = \Phi(x, f(x))$ is a linear bijection from $C^m(\mathbb{R}, c_{00})$ onto itself. However, there is a sequence of functions (f_n) in $C^m(\mathbb{R}, c_{00})$ so that $(f_n^{(k)})_n$ converges uniformly to 0 for all $k \leq m$, but $((Tf_n)^{(k)}(0))_n$ does not converge to 0 for any k, $1 \leq k \leq m$. Furthermore, the maps $\Phi_k(0, \cdot)$, $1 \leq k \leq m$, defined as in part (e) Theorem 23, are not bounded.*

If $f \in C^m(\mathbb{R}, c_{00})$, write $f = \sum f_i \mathbf{e}_i$, where (e_i) is the unit vector basis of c_{00} and $f_i \in C^m(\mathbb{R})$.

Lemma 25. *If* $f \in C^m(\mathbb{R}, c_{00})$, *then for all* $x_0 \in \mathbb{R}$ *and* $0 \leq k \leq m$, *there exists* $i_0 = i_0(x_0) \in \mathbb{N}$ *such that*

$$\lim_{x \to x_0} \sup_{i \geq i_0} \frac{|f_i^{(k)}(x)|}{|x - x_0|^{m-k}} = 0.$$

Proof. Let $x_0 \in \mathbb{R}$ and $0 \leq k \leq m$. Since $f \in C^m(\mathbb{R}, c_{00})$, for all $0 \leq j \leq m$,

$$(8) \qquad \lim_{x \to x_0} \sup_i |f_i^{(j)}(x) - f_i^{(j)}(x_0)| = 0.$$

Since $f^{(j)}(x_0) \in c_{00}$ for all $j \leq m$, there exists $i_0 \in \mathbb{N}$ such that $f_i^{(j)}(x_0) = 0$ for all $i \geq i_0$ and $0 \leq j \leq m$. For $i \geq i_0$, $x \neq x_0$, there exists, by Taylor's Theorem, $\xi = \xi(i, x)$ satisfying $0 < |\xi - x_0| < |x - x_0|$, such that

$$f_i^{(k)}(x) = f_i^{(k)}(x_0) + \cdots + \frac{f_i^{(m-1)}(x_0)}{(m-1-k)!}(x - x_0)^{m-1-k} + \frac{f_i^{(m)}(\xi)}{(m-k)!}(x - x_0)^{m-k}$$

$$= \frac{f_i^{(m)}(\xi)}{(m-k)!}(x - x_0)^{m-k}.$$

Thus

$$\frac{|f_i^{(k)}(x)|}{|x - x_0|^{m-k}} = \frac{|f_i^{(m)}(\xi)|}{(m-k)!}.$$

By (8), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{0 < |x - x_0| < \delta} \sup_{i \geq i_0} |f_i^{(m)}(x)| = \sup_{0 < |x - x_0| < \delta} \sup_{i \geq i_0} |f_i^{(m)}(x) - f_i^{(m)}(x_0)| < \varepsilon.$$

Therefore, for $0 < |x - x_0| < \delta$,

$$\sup_{i \geq i_0} \frac{|f_i^{(k)}(x)|}{|x - x_0|^{m-k}} < \frac{\varepsilon}{(m-k)!}.$$

$\square$

Lemma 26. *Let* (φ_i) *be a sequence in* $C^\infty(\mathbb{R})$. *If, for any* $x_0 \in \mathbb{R}$ *and* $0 \leq j \leq m$, *there exists* $i_0 \in \mathbb{N}$ *such that*

$$\limsup_{x \to x_0} \sup_{i \geq i_0} |\varphi_i^{(j)}(x)| |x - x_0|^j = M < \infty,$$

then $\sum f_i \varphi_i \mathbf{e}_i \in C^m(\mathbb{R}, c_{00})$ *whenever* $\sum f_i \mathbf{e}_i \in C^m(\mathbb{R}, c_{00})$.

Proof. Let $\varepsilon > 0$ and $0 \leq j, k \leq m$ with $j + k = m$. According to the hypothesis on (φ_i) and Lemma 25, there exists $\delta > 0$ and $i_1 \in \mathbb{N}$ such that

$$|\varphi_i^{(j)}(x)| |x - x_0|^j < M + 1 \text{ and } \frac{|f_i^{(k)}(x)|}{|x - x_0|^j} < \varepsilon$$

whenever $0 < |x - x_0| < \delta$ and $i \geq i_1$. Hence

$$\lim_{x \to x_0} \sup_{i \geq i_1} |f_i^{(k)}(x) \varphi_i^{(j)}(x)| = 0.$$

We may also assume that $f_i^{(\ell)}(x_0) = 0$ for $i \geq i_1$ and $0 \leq \ell \leq m$. Thus

$$\lim_{x \to x_0} \sup_{i \geq i_1} |f_i^{(k)}(x) \varphi_i^{(j)}(x) - f_i^{(k)}(x_0) \varphi_i^{(j)}(x_0)| = 0.$$

Since $f_i^{(k)} \varphi_i^{(j)}$ is continuous for $1 \leq i < i_1$,

$$\lim_{x \to x_0} \sup_i |f_i^{(k)}(x) \varphi_i^{(j)}(x) - f_i^{(k)}(x_0) \varphi_i^{(j)}(x_0)| = 0.$$

It follows that $\sum (f_i \varphi_i)^{(m)} \mathbf{e}_i \in C(\mathbb{R}, c_{00})$. $\qquad\qquad\square$

Proof of Example 24. Let φ be a C^∞-bump function. For all $j \in \mathbb{N} \cup \{0\}$, let $C_j = \left\| \varphi^{(j)} \right\|_\infty$. Set $\varphi_i(x) = i\varphi(ix) x^k + 2$ if $i \in \mathbb{N}_k = m\mathbb{N} + k, 1 \leq k \leq m$. Note that

$$(9) \qquad\qquad |\varphi(ix)| \leq \chi_{\left[-\frac{1}{i}, \frac{1}{i}\right]}(x).$$

In particular $\varphi_i(x) \geq 1$ for all x. Thus $\psi_i(x) = \frac{1}{\varphi_i(x)}$ is well-defined for $i \in \mathbb{N}$. We claim that for $x_0 \in \mathbb{R}$ and $0 \leq j \leq m$, there exists $i_0 \in \mathbb{N}$ such that

(1) $\limsup\limits_{x \to x_0} \sup_{i \geq i_0} |\varphi_i^{(j)}(x)| |x - x_0|^j < \infty$,

(2) $\limsup\limits_{x \to x_0} \sup_{i \geq i_0} |\psi_i^{(j)}(x)| |x - x_0|^j < \infty$.

If $x_0 \neq 0$, there exists i_0 and a neighborhood U of x_0 such that φ_i and ψ_i are constant on U for all $i > i_0$. Hence (1) and (2) hold. Assume that $x_0 = 0$. It follows from (9) that $1 \leq |\varphi_i(x)| \leq C_0 |x|^{k-1} + 2$ if $i \in \mathbb{N}_k$. Therefore (1) and (2) hold for $j = 0$. Let $1 \leq j \leq m$ be fixed. If $i \in \mathbb{N}_k$, then

$$\varphi_i^{(j)}(x) = i \sum_{\ell=0}^{j \wedge k} \binom{j}{\ell} \left[k(k-1) \cdots (k-\ell+1) x^{k-\ell} \right] \left[i^{j-\ell} \varphi^{(j-\ell)}(ix) \right].$$

Thus

$$|\varphi_i^{(j)}(x)| |x|^j \leq \sum_{\ell=0}^{j \wedge k} \binom{j}{\ell} k^\ell |x|^{k-1} \left| \varphi^{(j-\ell)}(ix) \right| |ix|^{j-\ell+1}$$

$$\leq m^m |x|^{k-1} \sum_{\ell=0}^{j \wedge k} \binom{j}{\ell} C_{j-\ell}.$$

The last inequality holds because $\varphi^{(j-\ell)}(ix) = 0$ when $|ix| > 1$. This proves that (1) is satisfied for $1 \leq j \leq m$.

We prove (2) by induction on j. Assume that (2) is satisfied for all $j \leq j_0$ where $j_0 \leq m - 1$. Note that

$$0 = (\varphi_i \psi_i)^{(j_0+1)} = \sum_{\ell=0}^{j_0+1} \binom{j_0+1}{\ell} \varphi_i^{(\ell)} \psi_i^{(j_0+1-\ell)}.$$

Therefore,

$$\psi_i^{(j_0+1)}(x) = \frac{-1}{\varphi_i(x)} \sum_{\ell=1}^{j_0+1} \binom{j_0+1}{\ell} \varphi_i^{(\ell)}(x) \psi_i^{(j_0+1-\ell)}(x).$$

Since $\left| \frac{1}{\varphi_i(x)} \right| \leq 1$,

$$|\psi_i^{(j_0+1)}(x)| |x|^{j_0+1} \leq \sum_{\ell=1}^{j_0+1} \binom{j_0+1}{\ell} |\varphi_i^{(\ell)}(x)| |x|^\ell |\psi_i^{(j_0+1-\ell)}(x)| |x|^{j_0+1-\ell}.$$

The desired inequality (2), with $j_0 + 1$ in place of j, follows from (1) and the inductive hypothesis.

By (1), (2) and Lemma 26, the map $T : C^m(\mathbb{R}, c_{00}) \to C^m(\mathbb{R}, c_{00})$, $\sum f_i \mathbf{e}_i \mapsto \sum \varphi_i f_i \mathbf{e}_i$ is a linear bijection. Finally, the sequence $(f_n) = \left(\frac{1}{n} \mathbf{e}_n \right)$ in $C^m(\mathbb{R}, c_{00})$ and the sequences of derivatives $(f_n^{(k)})_n$, $k \leq m$, converge to 0 uniformly. However, for $1 \leq k \leq m$, $(Tf_n)^{(k)}(0) = k! \mathbf{e}_n$ whenever $n \in \mathbb{N}_k$. Hence $\lim_{n \to \infty} \Phi_k \left(0, \frac{\mathbf{e}_n}{n} \right) = \lim_{n \to \infty} Tf_n^{(k)}(0) \neq 0$. In particular, $\Phi_k(0, \cdot)$ is not bounded. $\qquad\square$

Example 27. *There is a map $\Phi : \mathbb{R} \times c_{00} \to c_{00}$ such that the induced map T given by $Tf(x) = \Phi(x, f(x))$ is a linear bijection from $C^\infty(\mathbb{R}, c_{00})$ onto itself. However, $\Phi_k(0, \cdot)$ is not bounded for any $k \in \mathbb{N} \cup \{0\}$ and there is a sequence of functions (f_n) in $C^\infty(\mathbb{R}, c_{00})$ so that $(f_n^{(k)})_n$ converges uniformly to 0 for all $k \in \mathbb{N} \cup \{0\}$, but $((Tf_n)^{(k)}(0))$ does not converge to 0 for any $k \in \mathbb{N} \cup \{0\}$.*

Lemma 28. *Let (φ_i) be a sequence in $C^\infty(\mathbb{R})$. If for any $x_0 \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$, there exists $i_0 = i_0(k) \in \mathbb{N}$ such that*

$$\limsup_{x \to x_0} \sup_{i \geq i_0} |\varphi_i^{(k)}(x)| |x - x_0|^{k+1} < \infty,$$

then $\sum f_i \varphi_i \mathbf{e}_i \in C^\infty(\mathbb{R}, c_{00})$ whenever $\sum f_i \mathbf{e}_i \in C^\infty(\mathbb{R}, c_{00})$.

Proof. Given $x_0 \in \mathbb{R}$, $k, j \in \mathbb{N} \cup \{0\}$, it follows from Lemma 25 that there exists $i_1 = i_1(x_0, k, j) > i_0(j)$ such that $f_i^{(k)}(x_0) = 0$ if $i \geq i_0$ and

$$\lim_{x \to x_0} \sup_{i \geq i_1} \frac{|f_i^{(k)}(x)|}{|x - x_0|^{j+1}} = 0.$$

It follows readily that $\lim_{x \to x_0} \sup_{i \geq i_1} |f_i^{(k)}(x) \varphi_i^{(j)}(x)| = 0$. The remainder of the argument proceeds as in the proof of Lemma 26. $\qquad\square$

Proof of Example 27. Let φ be a C^∞-bump function. For all $j \in \mathbb{N} \cup \{0\}$, let $C_j = \| \varphi^{(j)} \|_\infty$. Let $(\mathbb{N}_k)_{k \in \mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets such that $i \geq k$ if $i \in \mathbb{N}_k$. Set $\varphi_i(x) = i\varphi(ix) x^{k-1} + 2$, if $i \in \mathbb{N}_k$ and $\psi_i(x) = \frac{1}{\varphi_i(x)}$. By arguments similar to those used in the proof of Example 24, one can show that for all $x_0 \in \mathbb{R}$ and $j \in \mathbb{N} \cup \{0\}$, there exists $i_0 \in \mathbb{N}$ such that

(1) $\lim_{x\to x_0} \sup_{i\geq i_0} |\varphi_i^{(j)}(x)| |x-x_0|^{j+1} < \infty$,

(2) $\lim_{x\to x_0} \sup_{i\geq i_0} |\psi_i^{(j)}(x)| |x-x_0|^{2j+1} < \infty$.

It follows by Lemma 28 that the map $T : C^\infty(\mathbb{R}, c_{00}) \to C^\infty(\mathbb{R}, c_{00})$, $\sum f_i \mathbf{e}_i \mapsto \sum \varphi_i f_i \mathbf{e}_i$, is a linear bijection. For each n, let $f_n = \frac{\mathbf{e}_n}{n} \in C^\infty(\mathbb{R}, c_{00})$. Then $\left(f_n^{(k)}\right)$ converges uniformly to zero for all $k \in \mathbb{N} \cup \{0\}$. However, $T f_n(x) = \dfrac{\varphi_n(x)\mathbf{e}_n}{n} = \varphi(nx) x^{k-1} + \frac{2}{n}$ for all $n \in \mathbb{N}_k$. $|x| < 1$. Recall that $\varphi_n(x) = 1$ if $|x| \leq \frac{1}{2n}$. Therefore, for all $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}_{k+1}$

$$(Tf_n)^{(k)}(0) = \begin{cases} 1 + 2/n & \text{if} \quad k = 0, \\ k! & \text{if} \quad k \geq 1. \end{cases}$$

Hence $(Tf_n)^{(k)}(0)$ does not converge to 0 for all k. Since $\Phi_k\left(0, \frac{e_n}{n}\right) = (Tf_n)^{(k)}(0)$, we also see that $\Phi_k(0, \cdot)$ is not bounded for any $k \in \mathbb{N} \cup \{0\}$. $\quad\square$

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